V_m^n "type" notation and Voigt or Mandel matrices

This section defines a special "V_m^n" tensor classification notation. Any tensor of type V_m^n will be seen to have m^n components. Furthermore, any tensor of type V_m^n is also of type V_N^1 where $N = m^n$. This latter identification helps us to convert indicial formulas involving 3×3 tensor matrices into matrix formulas involving 9×1 arrays. Similarly, indicial formulas involving $3 \times 3 \times 3$ third-order tensor matrices may be alternatively cast in terms of 3×9, 9×3 or even 27×1 arrays, depending on what is most convenient for actual computations using linear algebra libraries in modern computing environments.

Scalars are often called 0^{th}-order tensors. Vectors are sometimes called 1^{st}-order tensors. In general, an n^{th} order *engineering* tensor has 3^n components,* and we say that these tensors are of type V_3^n. For example, stress is a second-order tensor, so it has 3^2 (nine) components. Stress is symmetric, so these components are not all independent, but that doesn't change the fact that there are still nine components.

When solving a problem for which all tensors have symmetry with respect to some 2D plane embedded in 3D space, it is conventional to set up the basis so that the third basis vector points perpendicular to that plane. Doing this permits the 3D problem to be reduced to a 2D problem where vectors of interest now have only 2 nonzero (in-plane) components and second-order tensors are characterized by their (in-plane) 2×2 matrices, along with their lone 33 component. For example, biaxial stretching of a thin plate is nominally a 2D problem in which you only need to track the in-plane components of stress and strain. The out-of-plane 33 components can be analyzed separately, and all other components are zero. In a case like this, the pertinent tensors still have 3×3 component matrices and hence nine components — we are merely setting up a naturally aligned basis that allows some of those components to be consistently zero and hence ignorable.

When working in two dimensions, an n^{th}-order engineering tensor has 2^n components. Similarly, when working in an m-dimensional linear manifold (which is the higher dimensional version of a plane), an n^{th} order engineering tensor has m^n components, and we say it is of type V_m^n. The tensor's V_m^n type is highly pertinent to converting tensor expressions into matrix forms for actual calculations, so we now provide a summary of terminology and conventions for matrix versions of tensor equations.

* The base numeral 3 is used because of our emphasis on three-dimensional engineering contexts.

ALERT: The remainder of this section is rather abstract and can be skipped. It explains various ways to convert higher-order tensor operations into matrix forms needed to perform actual computations. Different matrix conventions are motivated by applicable constraints (such as symmetry) that might apply to tensors in the operations. Skip now to page 462 if you are a newcomer to tensor analysis. You have been warned!

A second-order tensor of type V_m^2 is *also* a first-order tensor of type V_N^1, where $N = m^2$. For example, an ordinary second-order engineering tensor (type V_3^2) is also a first-order vector in a 9-dimensional space (type V_9^1). Each has a total of 9 independent components, regardless of whether they are collected in the form of a 3×3 array or 9×1 array. Each of these nine components is associated with a basis tensor. With respect to the lab basis, for example, the matrix version of $\underset{\approx}{T} = T_{ij}\underset{\sim}{e}_i\underset{\sim}{e}_j$ might be written as

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = T_{11}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + T_{12}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + T_{13}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$+ T_{21}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + T_{22}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + T_{23}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$+ T_{31}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + T_{32}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + T_{33}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12.106}$$

In this case, the implied summation is expanded explicitly so that the 9 tensor components show up in the order

$$[T_{11}, T_{12}, T_{13}, T_{21}, T_{22}, T_{23}, T_{31}, T_{32}, T_{33}] \tag{12.107}$$

These components are paired with corresponding basis tensors,

$$\underset{\sim}{e}_1\underset{\sim}{e}_1,\ \underset{\sim}{e}_1\underset{\sim}{e}_2,\ \underset{\sim}{e}_1\underset{\sim}{e}_3,\ \underset{\sim}{e}_2\underset{\sim}{e}_1,\ \underset{\sim}{e}_2\underset{\sim}{e}_2,\ \underset{\sim}{e}_2\underset{\sim}{e}_3,\ \underset{\sim}{e}_3\underset{\sim}{e}_1,\ \underset{\sim}{e}_3\underset{\sim}{e}_2,\ \underset{\sim}{e}_3\underset{\sim}{e}_3, \tag{12.108}$$

which have lab component matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{12.109}$$

As with any basis, the ordering can always be changed. We could, for example, identify the 9 tensor components instead as

$$[T_{11}, T_{22}, T_{33}, T_{23}, T_{31}, T_{12}, T_{32}, T_{13}, T_{21}] \tag{12.110}$$

with associated basis tensors having component matrices given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{12.111}$$

This simple reordering of the basis tensors is a rudimentary form of a basis change for tensor space. Now let's do a non-trivial basis change.*

Just as you can speak of planes embedded in ordinary 3D space, you can also limit your attention to *subspaces* or *linear manifolds* contained within 9D tensor space. The set of all symmetric second-order engineering tensors, for example, is closed under tensor addition and scalar multiplication. By this we mean that any linear combination of symmetric tensors will be itself a symmetric tensor. Any symmetric engineering tensor (which has, at most, six independent components) can be therefore regarded as a *six*-dimensional vector belonging to a six-dimensional plane embedded in general nine-dimensional tensor space. Since symmetric engineering tensors belong to a linear manifold (a 6D plane), we say that they are of type V_6^1. Thus, symmetric second-order engineering tensors are *simultaneously* of type V_3^2, V_9^1, and V_6^1. It's a matter of how you want to interpret them for your problem, especially for matrix computations. A symmetric engineering tensor $\underset{\approx}{\mathbf{T}}$ still has 9 components, but 3 constraints ($T_{12} = T_{21}$, $T_{23} = T_{32}$, and $T_{31} = T_{13}$) exist among those components so that it has only 9–3=6 *independent* components. The goal of this chapter is to motivate a tensor basis change that is aligned with this type of symmetry.

An ordinary basis change to motivate the abstract one. This section discusses a simple 3D basis change from the lab basis to a basis that is aligned with a ramp. This discussion serves as a familiar precursor to what we will then do analogously as a relatively unfamiliar change of *tensor* basis.

Consider the set of ordinary 3D $\underset{\sim}{x}$-position vectors that are constrained to lie within a "tilted ramp" defined by the equation $x_2 = x_3$. This set is of type V_3^1 because each of the vectors still has three components, $\{x_1, x_2, x_3\}$, but it is also of type V_2^1 because the components are subjected to one constraint (namely x_2 and x_3 must be equal). Any linear combination of 3D vectors lying in this 2D plane will itself lie in the plane. Accordingly, the 2D tilted ramp is a linear manifold embedded within 3D space.

By saying that this set of vectors is both V_3^1 and V_2^1, we are implying that it is possible to set up a new orthonormal basis for which all vectors in the ramp have at most two nonzero components, while vectors perpendicular to the ramp have only one nonzero component. Let's get such a basis. The conventional laboratory expansion of a vector is

$$\underset{\sim}{x} = x_1 \underset{\sim}{e}_1 + x_2 \underset{\sim}{e}_2 + x_3 \underset{\sim}{e}_3 \,, \quad \text{which has matrix form} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \tag{12.112}$$

Knowing that we are interested in breaking up general vectors in to parts lying in the tilted ramp or perpendicular to this ramp, the above expansion could be written instead as

* As the discussion proceeds, keep in mind that we will be changing the basis used for *tensors* without changing any underlying ordinary 3D "laboratory" basis vectors $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$.

TENSOR ALGEBRA

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

$$\underset{\sim}{x} = x_1\underset{\sim}{e}_1 + \left(\frac{x_2+x_3}{2}\right)(\underset{\sim}{e}_2+\underset{\sim}{e}_3) + \left(\frac{x_3-x_2}{2}\right)(\underset{\sim}{e}_3-\underset{\sim}{e}_2),$$

which has matrix form
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \left(\frac{x_2+x_3}{2}\right)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \left(\frac{x_3-x_2}{2}\right)\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}. \tag{12.113}$$

By defining

$$x_{(2)} := \left(\frac{x_2+x_3}{2}\right), \quad x_{[3]} := \left(\frac{x_3-x_2}{2}\right),$$
$$\underset{\sim}{b}_1 = \underset{\sim}{e}_1, \quad \underset{\sim}{b}_2 = \underset{\sim}{e}_2+\underset{\sim}{e}_3, \quad \text{and} \quad \underset{\sim}{b}_3 = \underset{\sim}{e}_3-\underset{\sim}{e}_2, \tag{12.114}$$

the new expansion may be written a bit more compactly as

$$\underset{\sim}{x} = x_1\underset{\sim}{b}_1 + x_{(2)}\underset{\sim}{b}_2 + x_{[3]}\underset{\sim}{b}_3 \tag{12.115}$$

If the vector happens to fall on the tilted ramp where $x_2 = x_3$, the last term would drop out to give $\underset{\sim}{x} = x_1\underset{\sim}{b}_1 + x_{[2]}\underset{\sim}{b}_2$, making the vector *effectively* 2D even though it still has 3 components in the lab basis.

The expansion in Eq. (12.115) decomposes an arbitrary vector into a part contained within the ramp (where $x_2{=}x_3$) and a part perpendicular to it. The associated basis, $\{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\}$ is orthogonal — *but not orthonormal*. It is appealing because it is aligned with the ramp. For convenience of calculations, a rational engineer would normalize the basis to obtain yet another expansion:

$$\underset{\sim}{x} = x_1\hat{\underset{\sim}{e}}_1 + x_{\underline{2}}\hat{\underset{\sim}{e}}_2 + x_{\bar{3}}\hat{\underset{\sim}{e}}_3, \tag{12.116}$$

where

$$x_{\underline{2}} := \sqrt{2}x_{(2)} = \tfrac{\sqrt{2}}{2}(x_2+x_3), \qquad x_{\bar{3}} := \sqrt{2}x_{[3]} = \tfrac{\sqrt{2}}{2}(x_3-x_2),$$
$$\hat{\underset{\sim}{e}}_1 := \frac{\underset{\sim}{b}_1}{\|\underset{\sim}{b}_1\|} = \underset{\sim}{e}_1, \quad \hat{\underset{\sim}{e}}_2 := \frac{\underset{\sim}{b}_2}{\|\underset{\sim}{b}_2\|} = \frac{\underset{\sim}{e}_2+\underset{\sim}{e}_3}{\sqrt{2}}, \quad \text{and} \quad \hat{\underset{\sim}{e}}_3 := \frac{\underset{\sim}{b}_3}{\|\underset{\sim}{b}_3\|} = \frac{\underset{\sim}{e}_3-\underset{\sim}{e}_2}{\sqrt{2}}. \tag{12.117}$$

The act of dividing the last two basis vectors, $\underset{\sim}{b}_2$ and $\underset{\sim}{b}_3$, by their own $\sqrt{2}$ magnitudes has required multiplying the corresponding components by $\sqrt{2}$ so that the linear combination of components times basis vectors is unchanged. This new expansion has the matrix form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \sqrt{2}x_{(2)}\frac{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{2}} + \sqrt{2}x_{[3]}\frac{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}{\sqrt{2}}. \tag{12.118}$$

This form merely uses a new orthonormal basis that is aligned with the tilted ramp. Now let's go through a similar process for tensor space, where the analog of the 2D tilted ramp will be the 6D linear manifold (hyperplane) of symmetric tensors and the analog of the line perpendicular to the tilted ramp will be the 3D linear manifold of skew tensors.

Voigt and Mandel representations of second-order tensors. When considering symmetric tensors embedded within 9D tensor space, your calculations will often simplify if you switch away from the conventional laboratory $\underset{\sim}{e}_i\underset{\sim}{e}_j$ basis used in 9D tensor space to a different basis that is "aligned" with symmetric tensors. For example, instead of using $\underset{\sim}{e}_1\underset{\sim}{e}_2$ as one of your basis tensors, you might use $\underset{\sim}{e}_1\underset{\sim}{e}_2+\underset{\sim}{e}_2\underset{\sim}{e}_1$, which is symmetric. All of the other basis tensors would need to be redefined as well in order to switch to a basis that is aligned with symmetric tensors.

To see how this change-of-basis might be developed, note that the component expansion for the general (not necessarily symmetric) tensor in Eq. (12.106) can be written *equivalently* as

$$\begin{aligned}\begin{bmatrix}F_{11}&F_{12}&F_{13}\\F_{21}&F_{22}&F_{23}\\F_{31}&F_{32}&F_{33}\end{bmatrix} &= F_{11}\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}+F_{22}\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}+F_{33}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\\ &+ \left(\frac{F_{23}+F_{32}}{2}\right)\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}+\left(\frac{F_{31}+F_{13}}{2}\right)\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}+\left(\frac{F_{12}+F_{21}}{2}\right)\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}\\ &+ \left(\frac{F_{32}-F_{23}}{2}\right)\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}+\left(\frac{F_{13}-F_{31}}{2}\right)\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}+\left(\frac{F_{21}-F_{12}}{2}\right)\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}\end{aligned} \tag{12.119}$$

This equation is analogous to Eq. (12.113). Analogous to Eq. (12.114), we now define

$$F_{(ij)} = \tfrac{1}{2}(F_{ij}+F_{ji}) \qquad \text{and} \qquad F_{[ij]} = \tfrac{1}{2}(F_{ij}-F_{ji}), \tag{12.120}$$

Then Eq. (12.119) may be written more compactly as

$$\begin{aligned}\begin{bmatrix}F_{11}&F_{12}&F_{13}\\F_{21}&F_{22}&F_{23}\\F_{31}&F_{32}&F_{33}\end{bmatrix} &= F_{11}\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}+F_{22}\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}+F_{33}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}+F_{(23)}\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}+F_{(31)}\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}+F_{(12)}\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}\\ &+ F_{[32]}\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}+F_{[13]}\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}+F_{[21]}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}\end{aligned} \tag{12.121}$$

The nine matrices on the right side of this equation can be regarded as an alternative basis for 9D tensor space. The first six of these basis tensors are symmetric, and the last three are skew. This basis is still capable of describing arbitrary non-symmetric tensors. If $[F]$ happens to be symmetric, then $F_{23}=F_{32}$, $F_{31}=F_{13}$, and $F_{12}=F_{21}$, and the above expansion reduces to only the first six terms since the three $F_{[ij]}$ would all be zero. Thus, if you are dealing *exclusively* with symmetric tensors, then you only need the first six basis tensors. The components with respect to these particular basis tensors are called **Voigt components**. A disadvantage of the Voigt system is that only the first three basis tensors implied by Eq. (12.121) are *unit* tensors. The remaining ones have magnitude of $\sqrt{2}$.

The non-orthonormality of the Voigt basis prevents you from treating the Voigt array as if it were an ordinary vector (and thus precludes efficient use of vector functions built into modern computing environments). For example, the magnitude of a tensor is not found by the magnitude of its associated Voigt array. As was done in the analogous ramp-aligned-vector basis example, this inconvenience is easily remedied by simply normalizing the Voigt basis: each of the non-unit basis tensors is divided by its own magnitude, $\sqrt{2}$. As in the ramp analogy, dividing the basis tensors by $\sqrt{2}$ requires multiplying the coefficient by $\sqrt{2}$ to avoid changing the result of the summation. Specifically, defining

$$F_{\underline{ij}} = \tfrac{\sqrt{2}}{2}(F_{ij}+F_{ji}) \qquad \text{and} \qquad F_{\overline{ij}} = \tfrac{\sqrt{2}}{2}(F_{ij}-F_{ji}), \tag{12.122}$$

Eq. (12.121) may be written alternatively as

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} = F_{11}\begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix}^{\wedge} + F_{22}\begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix}^{\wedge} + F_{33}\begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix}^{\wedge} + F_{\underline{23}}\begin{bmatrix} 0&0&0\\0&0&1\\0&1&0 \end{bmatrix}^{\wedge} + F_{\underline{31}}\begin{bmatrix} 0&0&1\\0&0&0\\1&0&0 \end{bmatrix}^{\wedge} + F_{\underline{12}}\begin{bmatrix} 0&1&0\\1&0&0\\0&0&0 \end{bmatrix}^{\wedge}$$

$$+F_{\overline{32}}\begin{bmatrix} 0&0&0\\0&0&-1\\0&1&0 \end{bmatrix}^{\wedge} + F_{\overline{13}}\begin{bmatrix} 0&0&1\\0&0&0\\-1&0&0 \end{bmatrix}^{\wedge} + F_{\overline{21}}\begin{bmatrix} 0&-1&0\\1&0&0\\0&0&0 \end{bmatrix}^{\wedge} \tag{12.123}$$

Alert: the “hats” normalize!

The very important “hat” on the matrices denotes normalization (*i.e.*, division of the matrix by its own magnitude). This normalization of the Voigt basis produces the **Mandel basis**. The corresponding **Mandel components** are the same as Voigt components except that the components corresponding to off-diagonal positions within the original 3×3 matrix are multiplied by $\sqrt{2}$ in the Mandel 9×1 array. For example, the fourth Voigt component is $F_{(23)}$, whereas the fourth Mandel component is $F_{\underline{23}} = \sqrt{2}F_{(23)}$.

The change of basis to either the Voigt or Mandel system clearly shows that the set of all symmetric tensors needs only *six* basis tensors, making symmetric tensors of type V_6^1. Skew tensors are of type V_3^1, and we have defined the three skew-tensor components to coincide with “axial vector” components defined later in Eq. (13.29).

In engineering mechanics, Voigt arrays for symmetric tensors most commonly arise in stress-strain constitutive models, where phrases like “strain-like” or “stress-like” are used for what should be properly termed as *covariant* and *contravariant*, respectively. For symmetric tensors, these are defined as follows:

Covariant “strain-like” Voigt array for a symmetric tensor $\underset{\approx}{\varepsilon}$:

$$[\varepsilon_1^V, \varepsilon_2^V, \varepsilon_3^V, \varepsilon_4^V, \varepsilon_5^V, \varepsilon_6^V] = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{31}, 2\varepsilon_{12}] \tag{12.124a}$$

These are paired with *contravariant* basis tensors having lab components given by

$$[\underset{\approx}{G}_V^1],\ [\underset{\approx}{G}_V^2],\ [\underset{\approx}{G}_V^3],\ [\underset{\approx}{G}_V^4],\ [\underset{\approx}{G}_V^5],\ [\underset{\approx}{G}_V^6] =$$

$$\begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix}, \frac{\begin{bmatrix} 0&0&0\\0&0&1\\0&1&0 \end{bmatrix}}{2}, \frac{\begin{bmatrix} 0&0&1\\0&0&0\\1&0&0 \end{bmatrix}}{2}, \frac{\begin{bmatrix} 0&1&0\\1&0&0\\0&0&0 \end{bmatrix}}{2}. \tag{12.124b}$$

The expansion, $\underset{\approx}{\boldsymbol{\varepsilon}} = \varepsilon_{ij}\underset{\sim}{\boldsymbol{e}}_i\underset{\sim}{\boldsymbol{e}}_j$, may be then written as $\underset{\approx}{\boldsymbol{\varepsilon}} = \sum_{I=1}^{9} \varepsilon_I^V \underset{\approx}{\boldsymbol{G}}_V^I$. (12.124c)

Contravariant "stress-like" Voigt array for a symmetric tensor $\underset{\approx}{\boldsymbol{\sigma}}$:

$$[\sigma_V^1, \sigma_V^2, \sigma_V^3, \sigma_V^4, \sigma_V^5, \sigma_V^6] = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}] \tag{12.125a}$$

These are paired with *covariant* basis tensors having lab components given by

$$[\underset{\approx}{\boldsymbol{G}}_1^V],\ [\underset{\approx}{\boldsymbol{G}}_2^V],\ [\underset{\approx}{\boldsymbol{G}}_3^V],\ [\underset{\approx}{\boldsymbol{G}}_4^V],\ [\underset{\approx}{\boldsymbol{G}}_5^V],\ [\underset{\approx}{\boldsymbol{G}}_6^V] =$$

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix} \tag{12.125b}$$

The expansion, $\underset{\approx}{\boldsymbol{\sigma}} = \sigma_{ij}\underset{\sim}{\boldsymbol{e}}_i\underset{\sim}{\boldsymbol{e}}_j$, may be then written as $\underset{\approx}{\boldsymbol{\sigma}} = \sum_{I=1}^{9} \sigma_V^I \underset{\approx}{\boldsymbol{G}}_I^V$. (12.125c)

Consistent with standard notation for non-orthonormal basis systems, covariant indices are subscripts (for which the mnemonic "co-go-below" is helpful). The contravariant indices are superscripts. Such conventions are needed only when the basis is not orthonormal. For orthonormal bases, like the Mandel system, all indices are subscripts. Thus,

Mandel array for a symmetric tensor $\underset{\approx}{\boldsymbol{T}}$:

$$[T_1, T_2, T_3, T_4, T_5, T_6] = [T_{11}, T_{22}, T_{33}, \sqrt{2}T_{23}, \sqrt{2}T_{31}, \sqrt{2}T_{12}] \tag{12.126a}$$

These are paired with *orthonormal* basis tensors having lab components given by

$$[\underset{\approx}{\boldsymbol{e}}_1],\ [\underset{\approx}{\boldsymbol{e}}_2],\ [\underset{\approx}{\boldsymbol{e}}_3],\ [\underset{\approx}{\boldsymbol{e}}_4],\ [\underset{\approx}{\boldsymbol{e}}_5],\ [\underset{\approx}{\boldsymbol{e}}_6] =$$

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \frac{\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}}{\sqrt{2}}, \frac{\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}}{\sqrt{2}}, \frac{\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}}{\sqrt{2}}. \tag{12.126b}$$

The expansion, $\underset{\approx}{\boldsymbol{T}} = T_{ij}\underset{\sim}{\boldsymbol{e}}_i\underset{\sim}{\boldsymbol{e}}_j$, may be then written as $\underset{\approx}{\boldsymbol{T}} = \sum_{I=1}^{9} T_I \underset{\approx}{\boldsymbol{e}}_I$. (12.126c)

Advantages of the Mandel system become apparent when you start doing calculations. For example, the magnitude of a tensor, $\underset{\approx}{\boldsymbol{A}}$, is defined in terms of conventional components by

$$\|\underset{\approx}{\boldsymbol{A}}\| = \sqrt{\underset{\approx}{\boldsymbol{A}}:\underset{\approx}{\boldsymbol{A}}} = \sqrt{A_{ij}A_{ij}}$$
$$= \sqrt{A_{11}^2 + A_{22}^2 + A_{33}^2 + A_{23}^2 + A_{31}^2 + A_{12}^2 + A_{32}^2 + A_{13}^2 + A_{21}^2} \tag{12.127}$$

If the tensor is symmetric, the off-diagonal terms combine to give

$$\|\underset{\approx}{\boldsymbol{A}}\| = \sqrt{A_{11}^2 + A_{22}^2 + A_{33}^2 + \mathbf{2}\,A_{23}^2 + \mathbf{2}\,A_{31}^2 + \mathbf{2}\,A_{12}^2}. \tag{12.128}$$

When computed using contravariant (stress-like) Voigt components, this operation is

$$\|\underset{\approx}{\boldsymbol{A}}\| = \sqrt{(A_V^1)^2 + (A_V^2)^2 + (A_V^3)^2 + \mathbf{2}\,(A_V^4)^2 + \mathbf{2}\,(A_V^5)^2 + \mathbf{2}\,(A_V^6)^2} \tag{12.129}$$

When computed using covariant (strain-like) components, the same operation is

$$\|\underset{\approx}{\mathbf{A}}\| = \sqrt{(A_1^V)^2 + (A_2^V)^2 + (A_3^V)^2 + \tfrac{1}{2}(A_4^V)^2 + \tfrac{1}{2}(A_5^V)^2 + \tfrac{1}{2}(A_6^V)^2}\,. \tag{12.130}$$

When computed using Mandel components, on the other hand, the magnitude of the tensor identically equals the magnitude of its Mandel vector array:

$$\|\underset{\approx}{\mathbf{A}}\| = \sqrt{A_1^2 + A_2^2 + A_3^2 + A_4^2 + A_5^2 + A_6^2} \tag{12.131}$$

Such intoxicating simplicity is lost when using a Voigt system, which requires jolting factors or divisors of 2.

Some moderately recent discourses on Mandel components and their associated basis can be found in Refs. [48, 94, 57, 67], but publications that use the $\sqrt{2}$ in component arrays usually don't give a name to the approach (and often don't even explain that the components are the result of a tensor basis change). We have chosen the name "Mandel" after one of the earliest researchers that we found to extensively use the factor of $\sqrt{2}$ (*cf.* ___).

Voigt and Mandel representations of fourth-order tensors. Now that we have established that a second-order (type V_3^2) tensor can be regarded as a nine-dimensional (type V_9^1) vector with a corresponding 9×1 component array, it follows that fourth-order (type V_3^4) tensors can be viewed as (type V_9^2) second-order tensors with respect to a 9D space. Consequently, they can be manipulated in computations by using a 9×9 matrix, with the indices ranging from 1 to 9 corresponding to index pairs selected above for 9D Voigt or Mandel representations of second-order tensors: 11, 22, 33, 23, 31, 12, 32, 13, 21. If you limit attention to fourth-order tensors that are minor symmetric (which means $A_{ijkl} = A_{jikl} = A_{ijlk}$), then you would be well-advised to abandon the standard lab dyad basis used in Eq. (12.38) in favor of the alternative basis tensors listed in Eq. (12.119). When using that basis, the last three columns and last three rows of a 9×9 matrix for a minor-symmetric fourth-order tensor will contain *all zeros*. In other words, you will be dealing only with the upper 6×6 part of the matrix. Consequently, minor-symmetric fourth-order tensors are of type V_6^2 and they have at most 6^2, or 36, nonzero components. The 6×6 Mandel component matrix is the same as the 6×6 Voigt matrix except that the last three rows and columns are multiplied by $\sqrt{2}$ (making the lower right 3×3 submatrix ultimately multiplied by 2). The Mandel matrix will be symmetric if the original fourth-order tensor is major symmetric.

Being orthonormalized, the Mandel basis tends to produce more intuitive component matrices than the Voigt system. For example, the fourth-order identity tensor for symmetric matrices (called $\underset{\approx}{\mathbf{P}}^{\text{sym}}$ later in this book) maps to the 6×6 identity matrix under the Mandel system (but not with the Voigt system). Also, the concept of an eigenvector, found using standard numerical eigensolvers, applies to Mandel representations but not Voigt. A standard eigenvector of a Voigt matrix is physically meaningless. The Mandel system is relatively intuitive because it uses an orthonormal basis for tensors. As explained in

Ref. [14], Voigt component formulas typically contain surprising factors or divisors of 2 or 4 because the underlying Voigt basis is not orthonormal. According to parlance of non-orthonormal basis (cf. [14]), Voigt component arrays have **covariant**, **contravariant**, and (for higher-order tensors) **mixed** forms.

Voigt and Mandel notation for 2nd-order tensors. An ability to change how you regard the "V_m^n type" of a second-order tensor is useful in materials mechanics. For example, in plasticity, the trial elastic stress rate is found by assuming that a material is behaving elastically under strain-controlled loading. If it is found that this assumption would move the stress into a "forbidden" region that violates the yield condition, then plastic flow must be occurring. The set of admissible *elastic* stresses is defined by a yield function such that $f(\underset{\approx}{\sigma}) < 0$. When $\underset{\approx}{\sigma}$ is regarded as a *vector* of type V_6^1, then $f(\underset{\approx}{\sigma}) = 0$ defines a yield *surface* in 6D space. For example, just as the equation $\underset{\sim}{x} \bullet \underset{\sim}{x} - R^2 = 0$ defines a sphere of radius R in ordinary 3D space, the equation $\underset{\approx}{\sigma} : \underset{\approx}{\sigma} - R^2 = 0$ would define a *hypersphere* in 6D stress space. When the trial assumption of elastic behavior is found to move the stress into inadmissible stress states (i.e., those for which $f(\underset{\approx}{\sigma}) > 0$), then the equations governing classical nonhardening plasticity can be used to show that the *actual* stress rate is obtained by projecting the trial elastic stress rate onto the yield surface [16]. The projection operation in plasticity is similar in structure to the projection shown in Fig. 11.4 except that the vector dot product is replaced by the tensor inner product ($:$). The outward "normal" $\underset{\approx}{B}$ that defines the target plane is the gradient of the yield function (i.e., $B_{ij} = \partial f / \partial \sigma_{ij}$). Statements like this implicitly regard tensors as *vectors* (higher dimension vectors of type V_6^1). This determination of the yield surface normal direction is perfectly analogous to finding the normal to an ordinary 3D surface by taking the spatial gradient of its defining level-set function $f(\underset{\sim}{x})$.

Examples of Voigt and Mandel notation for 3rd-order tensors.

This section revisits two previously analyzed transformations relating 1^{st}-order and 2^{nd}-order tensors, each characterized by a 3^{rd}-order tensor. Conventionally, a 3^{rd} order tensor has a $3 \times 3 \times 3$ component array. We can consider this instead as a 3×9 matrix when the transformation input is a second-order tensor (which has 9 components). We can consider the $3 \times 3 \times 3$ component array as a 9×3 matrix when the input is a vector. As illustrated in these examples, the matrix may be reduced to 3×6 or 6×3 if the tensor is symmetric.

<u>EXAMPLE 1</u>: Transformation of a second-order tensor to a vector $(V_3^2 \to V_3^1)$.

Consider a relationship of the form $\underset{\sim}{v} = \underset{\approx}{A} \bullet \underset{\sim}{c}$. This way of writing the formula emphasizes that the vector $\underset{\sim}{v}$ depends linearly on the vector $\underset{\sim}{c}$, but it also depends linearly on the tensor $\underset{\approx}{A}$. Recalling the operator Y_{ijk} defined in Eq. (12.97), we may also write

$$\underset{\approx}{A} \bullet \underset{\sim}{c} = \underset{\approx}{Y} : \underset{\approx}{A}, \qquad \text{where } Y_{ijk} = \delta_{ij} c_k \tag{12.132}$$

The third-order tensor $\underset{\approx}{Y}$ is introduced to emphasize that the vector $\underset{\sim}{v} = \underset{\approx}{A} \bullet \underset{\sim}{c}$ depends linearly on the tensor $\underset{\approx}{A}$, allowing the operation to be written as $\underset{\sim}{v} = \underset{\approx}{Y} : \underset{\approx}{A}$. This is the standard symbolic (direct) notation for a vector-valued linear transformation of a tensor. We desire now to express this formula in standard matrix form (where the motivation might be, for example, to exploit computational linear algebra libraries that are available in modern computing environments). We also aim to find a matrix formulation that is computationally convenient for the special case that $\underset{\approx}{A}$ is symmetric, which allows Y_{ijk} to be replaced without loss in generality by the formula in Eq. (12.98),

$$Y_{i(jk)} = \tfrac{1}{2}(\delta_{ij} c_k + \delta_{ik} c_j). \tag{12.133}$$

Wrapping the last two indices (jk) in parentheses indicates imposed symmetry with respect to this index pair. For now, we will presume that $\underset{\approx}{A}$ is potentially nonsymmetric. Then:

Explicit form of $\underset{\sim}{v} = \underset{\approx}{A} \bullet \underset{\sim}{c}$:

$$v_1 = A_{11}c_1 + A_{12}c_2 + A_{13}c_3 \tag{12.134a}$$

$$v_2 = A_{21}c_1 + A_{22}c_2 + A_{23}c_3 \tag{12.134b}$$

$$v_3 = A_{31}c_1 + A_{32}c_2 + A_{33}c_3 \tag{12.134c}$$

If this is interpreted as a linear operator for which $\underset{\approx}{A}$ is the input,

$$\text{Matrix form of } \underset{\sim}{v} = \underset{\approx}{A} \bullet \underset{\sim}{c}: \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 & c_2 & 0 & c_3 & 0 \\ 0 & c_2 & 0 & c_3 & 0 & 0 & 0 & 0 & c_1 \\ 0 & 0 & c_3 & 0 & c_1 & 0 & c_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ A_{23} \\ A_{31} \\ A_{12} \\ A_{32} \\ A_{13} \\ A_{21} \end{bmatrix}. \tag{12.135}$$

This is a special case of a more generic statement:

$$\text{Matrix form of } \underset{\sim}{v} = \underset{\approx}{Y}{:}\underset{\approx}{A}: \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} Y_{111} & Y_{122} & Y_{133} & Y_{123} & Y_{131} & Y_{112} & Y_{132} & Y_{113} & Y_{121} \\ Y_{211} & Y_{222} & Y_{233} & Y_{223} & Y_{231} & Y_{212} & Y_{232} & Y_{213} & Y_{221} \\ Y_{311} & Y_{322} & Y_{333} & Y_{323} & Y_{331} & Y_{312} & Y_{332} & Y_{313} & Y_{321} \end{bmatrix} \begin{bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ A_{23} \\ A_{31} \\ A_{12} \\ A_{32} \\ A_{13} \\ A_{21} \end{bmatrix} \tag{12.136}$$

The 3×9 matrix in Eq. (12.135) is the $V_3^1 \times V_9^1$ component representation of the third-order tensor $\underset{\approx}{Y}$ for which $\underset{\sim}{v} = \underset{\approx}{A} \bullet \underset{\sim}{c}$ is re-written in the form $\underset{\sim}{v} = \underset{\approx}{Y}{:}\underset{\approx}{A}$. Comparing the corresponding matrix equations shows, for example, that $Y_{113} = c_3$, which is, of course, the same as what is implied from the indicial formula $Y_{ijk} = \delta_{ij} c_k$ derived in Eq. (12.97).

By expansion, you can confirm that Eq. (12.136) may be written alternatively as

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} Y_{111} & Y_{122} & Y_{133} & 2Y_{1(23)} & 2Y_{1(31)} & 2Y_{1(12)} & 2Y_{1[32]} & 2Y_{1[13]} & 2Y_{1[21]} \\ Y_{211} & Y_{222} & Y_{233} & 2Y_{2(23)} & 2Y_{2(31)} & 2Y_{2(12)} & 2Y_{2[32]} & 2Y_{2[13]} & 2Y_{2[21]} \\ Y_{311} & Y_{322} & Y_{333} & 2Y_{3(23)} & 2Y_{3(31)} & 2Y_{3(12)} & 2Y_{3[32]} & 2Y_{3[13]} & 2Y_{3[21]} \end{bmatrix} \begin{bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ A_{(23)} \\ A_{(31)} \\ A_{(12)} \\ A_{[32]} \\ A_{[13]} \\ A_{[21]} \end{bmatrix} \tag{12.137}$$

where

$$Y_{i(jk)} = \tfrac{1}{2}(Y_{ijk} + Y_{ikj}) \qquad \text{and} \qquad A_{(ij)} = \tfrac{1}{2}(A_{ij} + A_{ji}) \tag{12.138a}$$

$$Y_{i[jk]} = \tfrac{1}{2}(Y_{ijk} - Y_{ikj}) \qquad \text{and} \qquad A_{[ij]} = \tfrac{1}{2}(A_{ij} - A_{ji}) \tag{12.138b}$$

Applying Eq. (12.137) to the special case of Eq. (12.135) gives:

$$\text{New matrix form of } \underset{\sim}{v} = \underset{\approx}{A} \bullet \underset{\sim}{c}: \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & 0 & 0 & c_3 & c_2 & 0 & c_3 & -c_2 \\ 0 & c_2 & 0 & c_3 & 0 & c_1 & 0 & 0 & c_1 \\ 0 & 0 & c_3 & c_2 & c_1 & 0 & c_2 & -c_1 & 0 \end{bmatrix} \begin{bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ A_{(23)} \\ A_{(31)} \\ A_{(12)} \\ A_{[32]} \\ A_{[13]} \\ A_{[21]} \end{bmatrix}. \tag{12.139}$$

If the tensor $\underset{\approx}{A}$ is symmetric, the last three components of its 9×1 array in this formula are zero (i.e., $A_{[ij]} = 0$), which allows these explicit formulas to be written alternatively in reduced matrix form as

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \underbrace{\begin{bmatrix} c_1 & 0 & 0 & 0 & c_2 & c_3 \\ 0 & c_2 & 0 & c_3 & 0 & c_1 \\ 0 & 0 & c_3 & c_2 & c_1 & 0 \end{bmatrix}}_{\text{covariant Voigt matrix}} \overbrace{\begin{bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ A_{(23)} \\ A_{(31)} \\ A_{(12)} \end{bmatrix}}^{\text{contravariant Voigt array}} = \underbrace{\begin{bmatrix} Y_{111} & Y_{122} & Y_{133} & 2Y_{1(23)} & 2Y_{1(31)} & 2Y_{1(12)} \\ Y_{211} & Y_{222} & Y_{233} & 2Y_{2(23)} & 2Y_{2(31)} & 2Y_{2(12)} \\ Y_{311} & Y_{322} & Y_{333} & 2Y_{3(23)} & 2Y_{3(31)} & 2Y_{3(12)} \end{bmatrix}}_{\text{covariant Voigt matrix}} \overbrace{\begin{bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ A_{(23)} \\ A_{(31)} \\ A_{(12)} \end{bmatrix}}^{\text{contravariant Voigt array}}, \qquad (12.140a)$$

We refer to each 3×6 matrix in this formula as the **covariant Voigt** representation of the third-order tensor $\underset{\approx}{Y}$, applicable to the special case that $\underset{\approx}{A}$ is symmetric. An alternative representation moves the factors of 2 to a different location in such a manner that the result of the matrix multiplication is unchanged:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \underbrace{\begin{bmatrix} c_1 & 0 & 0 & 0 & \frac{c_2}{2} & \frac{c_3}{2} \\ 0 & c_2 & 0 & \frac{c_3}{2} & 0 & \frac{c_1}{2} \\ 0 & 0 & c_3 & \frac{c_2}{2} & \frac{c_1}{2} & 0 \end{bmatrix}}_{\text{contravariant Voigt matrix}} \overbrace{\begin{bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ 2A_{(23)} \\ 2A_{(31)} \\ 2A_{(12)} \end{bmatrix}}^{\text{covariant Voigt array}} = \underbrace{\begin{bmatrix} Y_{111} & Y_{122} & Y_{133} & Y_{1(23)} & Y_{1(31)} & Y_{1(12)} \\ Y_{211} & Y_{222} & Y_{233} & Y_{2(23)} & Y_{2(31)} & Y_{2(12)} \\ Y_{311} & Y_{322} & Y_{333} & Y_{3(23)} & Y_{3(31)} & Y_{3(12)} \end{bmatrix}}_{\text{contravariant Voigt matrix}} \overbrace{\begin{bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ 2A_{(23)} \\ 2A_{(31)} \\ 2A_{(12)} \end{bmatrix}}^{\text{covariant Voigt array}}, \qquad (12.140b)$$

We refer to each 3×6 matrix in this formula as the **contravariant Voigt** representation of the third-order tensor $\underset{\approx}{Y}$, again applicable to the special case that $\underset{\approx}{A}$ is symmetric.

With Voigt notation, each matrix multiplication must be between one covariant and one contravariant representation (i.e., the covariant factor must include the **metric** multiplier of 2 and the contravariant factor must omit it). The metric arises because the Voigt basis is not normalized. The Mandel representation for second (and higher) order tensors eliminates the confusing metrics by normalizing the basis for second-order tensors, so there is no need to distinguish between contravariant and covariant components. In particular, the Mandel revision of Eq. (12.137) is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \underbrace{\begin{bmatrix} Y_{111} & Y_{122} & Y_{133} & Y_{1\underline{23}} & Y_{1\underline{31}} & Y_{1\underline{12}} & Y_{1\overline{32}} & Y_{1\overline{13}} & Y_{1\overline{21}} \\ Y_{211} & Y_{222} & Y_{233} & Y_{2\underline{23}} & Y_{2\underline{31}} & Y_{2\underline{12}} & Y_{2\overline{32}} & Y_{2\overline{13}} & Y_{2\overline{21}} \\ Y_{311} & Y_{322} & Y_{333} & Y_{3\underline{23}} & Y_{3\underline{31}} & Y_{3\underline{12}} & Y_{3\overline{32}} & Y_{3\overline{13}} & Y_{3\overline{21}} \end{bmatrix}}_{\text{Mandel matrix}} \overbrace{\begin{bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ A_{\underline{23}} \\ A_{\underline{31}} \\ A_{\underline{12}} \\ A_{\overline{32}} \\ A_{\overline{13}} \\ A_{\overline{21}} \end{bmatrix}}^{\text{Mandel array}} \qquad (12.141)$$

where

$$Y_{i\underline{jk}} = \sqrt{2}\,Y_{i(jk)} = \frac{\sqrt{2}}{2}(Y_{ijk} + Y_{ikj}) \quad \text{and} \quad A_{\underline{ij}} = \sqrt{2}A_{(ij)} = \frac{\sqrt{2}}{2}(A_{ij} + A_{ji}) \qquad (12.142a)$$

$$Y_{i\overline{jk}} = \sqrt{2}\,Y_{i[jk]} = \frac{\sqrt{2}}{2}(Y_{ijk} - Y_{ikj}) \quad \text{and} \quad A_{\overline{ij}} = \sqrt{2}A_{[ij]} = \frac{\sqrt{2}}{2}(A_{ij} - A_{ji}) \qquad (12.142b)$$

By avoiding the confusion of two different component types (contravariant and covariant), the Mandel system is much easier to work with. To create a Mandel matrix, simply multiply the Voigt contravariant components by $\sqrt{2}$. In the case of symmetry, the size of the matrices may be reduced as previously discussed for Voigt notation. In this example, therefore,

Mandel matrix:

$$\begin{bmatrix} Y_{111} & Y_{122} & Y_{133} & Y_{1\underline{23}} & Y_{1\underline{31}} & Y_{1\underline{12}} \\ Y_{211} & Y_{222} & Y_{233} & Y_{2\underline{23}} & Y_{2\underline{31}} & Y_{2\underline{12}} \\ Y_{311} & Y_{322} & Y_{333} & Y_{3\underline{23}} & Y_{3\underline{31}} & Y_{3\underline{12}} \end{bmatrix} = \begin{bmatrix} c_1 & 0 & 0 & 0 & \frac{c_2}{\sqrt{2}} & \frac{c_3}{\sqrt{2}} \\ 0 & c_2 & 0 & \frac{c_3}{\sqrt{2}} & 0 & \frac{c_1}{\sqrt{2}} \\ 0 & 0 & c_3 & \frac{c_2}{\sqrt{2}} & \frac{c_1}{\sqrt{2}} & 0 \end{bmatrix} \qquad (12.143)$$

To obtain Voigt contravariant components, just divide the Mandel components by $\sqrt{2}$. The reason for forming a matrix representation in the first place is typically to facilitate efficient computations — the Mandel system serves this purpose far better than Voigt components. Not only does the Mandel system treat off-diagonal components in the same way (no worries about whether or not to use a factor of 2), it also allows *meaningful* use of standard linear algebra packages such as singular-value decomposition.

Equations (12.140a) and (12.140b) are different matrix-notation formulations for the *single* direct-notation formula,

$$\underset{\sim}{v} = \underset{\approx}{A} \bullet \underset{\sim}{c} = \underset{\approx}{Y} : \underset{\approx}{A} \qquad (12.144a)$$

The conventional indicial form for this equation is

$$v_i = Y_{ijk}A_{jk}, \qquad (12.144b)$$

where each index ranges from 1 to 3 and repeated indices are implicitly summed.

Recalling the Voigt components and basis tensors defined in Eqs. (12.124b) and (12.125b), the indicial-notation versions of Eqs. (12.140a) and (12.140b) are respectively

Voigt indicial form for a contravariant input: $\quad v_i = Y_{i\mathcal{J}}^{\;\;\mathrm{V}} A^{\mathcal{J}}_{\mathrm{V}}$ (12.144c)

Voigt indicial form for a covariant input: $\quad v_i = Y_{i\mathrm{V}}^{\;\;\mathcal{J}} A^{\mathrm{V}}_{\mathcal{J}}$ (12.144d)

The index i ranges from 1 to 3, while the repeated index $\mathcal{J}$ is implicitly summed from 1 to 6. The label "V" is just indicating that these are Voigt components (it isn't a summed index). Recalling the normalized basis and associated component definitions used in the Mandel system (where there is no need to distinguish between contravariant and covariant components), the Mandel indicial formula corresponding to Eq. (12.141) is simply

Mandel indicial form: $v_i = Y_{i\mathcal{J}} A_{\mathcal{J}}$ (12.144b)

As with the Voigt system, the free index i varies from 1 to 4, while the repeated index $\mathcal{J}$ is summed from 1 to 9. Unlike the Voigt system, the Mandel system uses an orthonormal basis and hence doesn't need distinction between covariant and contravariant components.

EXAMPLE 2: Transformation of a vector to a tensor $(V_3^1 \to V_3^2)$.

Regarding $\underset{\sim}{c}$ as a known constant vector, it was proved on page 331 that an operation $\underset{\approx}{B} = \frac{1}{2}(\underset{\sim}{c}\underset{\sim}{u} + \underset{\sim}{u}\underset{\sim}{c})$ may be written as a linear transformation of $\underset{\sim}{u}$. Namely,

$$\underset{\approx}{B} = \underset{\approx}{Z} \bullet \underset{\sim}{u}, \quad \text{which means } B_{ij} = Z_{ijk}u_k \quad \text{with} \quad Z_{ijk} = \tfrac{1}{2}(\delta_{kj}c_i + \delta_{ki}c_j) \tag{12.145}$$

The input to the operation $\underset{\approx}{B} = \underset{\approx}{Z} \bullet \underset{\sim}{u}$ is a vector $\underset{\sim}{u}$, while the output is the symmetric tensor $\underset{\approx}{B} = \frac{1}{2}(\underset{\sim}{c}\underset{\sim}{u} + \underset{\sim}{u}\underset{\sim}{c})$. This symmetry implies that $Z_{ijk} = Z_{(ij)k}$. Accordingly, we should represent $\underset{\approx}{Z}$ as a 6×3 matrix. The Voigt representations of this matrix are

Covariant Voigt:
$$\begin{bmatrix} Z_{111} & Z_{112} & Z_{311} \\ Z_{221} & Z_{222} & Z_{322} \\ Z_{331} & Z_{332} & Z_{333} \\ 2Z_{(23)1} & 2Z_{(23)2} & 2Z_{(23)3} \\ 2Z_{(31)1} & 2Z_{(31)2} & 2Z_{(31)3} \\ 2Z_{(12)1} & 2Z_{(12)2} & 2Z_{(12)3} \end{bmatrix} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \\ 0 & c_3 & c_2 \\ c_2 & 0 & c_1 \\ c_3 & c_1 & 0 \end{bmatrix} \tag{12.146}$$

Contravariant Voigt:
$$\begin{bmatrix} Z_{111} & Z_{112} & Z_{311} \\ Z_{221} & Z_{222} & Z_{322} \\ Z_{331} & Z_{332} & Z_{333} \\ Z_{(23)1} & Z_{(23)2} & Z_{(23)3} \\ Z_{(31)1} & Z_{(31)2} & Z_{(31)3} \\ Z_{(12)1} & Z_{(12)2} & Z_{(12)3} \end{bmatrix} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \\ 0 & \frac{c_3}{2} & \frac{c_2}{2} \\ \frac{c_2}{2} & 0 & \frac{c_1}{2} \\ \frac{c_3}{2} & \frac{c_1}{2} & 0 \end{bmatrix} \tag{12.147}$$

The Mandel representation is constructed by multiplying the contravariant Voigt matrix by a factor $\sqrt{2}$ at the locations corresponding to off-diagonal `$(ij)$` indices:

Mandel:
$$\begin{bmatrix} Z_{111} & Z_{112} & Z_{311} \\ Z_{221} & Z_{222} & Z_{322} \\ Z_{331} & Z_{332} & Z_{333} \\ Z_{\underline{23}1} & Z_{\underline{23}2} & Z_{\underline{23}3} \\ Z_{\underline{31}1} & Z_{\underline{31}2} & Z_{\underline{31}3} \\ Z_{\underline{12}1} & Z_{\underline{12}2} & Z_{\underline{12}3} \end{bmatrix} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \\ 0 & \frac{c_3}{\sqrt{2}} & \frac{c_2}{\sqrt{2}} \\ \frac{c_2}{\sqrt{2}} & 0 & \frac{c_1}{\sqrt{2}} \\ \frac{c_3}{\sqrt{2}} & \frac{c_1}{\sqrt{2}} & 0 \end{bmatrix} \tag{12.148}$$

This 6×1 representation of $\underset{\approx}{\mathbf{Z}}$ is the transpose of the 1×6 component matrix in Eq. (12.143) for the tensor $\underset{\approx}{\mathbf{Y}}$ of the previous example! This observation exploited in applications of small-deformation mechanics (see page 1043).

In the covariant Voigt representation, matrix and indicial forms of $\underset{\approx}{\mathbf{B}} = \underset{\approx}{\mathbf{Z}} \bullet \underset{\sim}{\mathbf{u}}$ are

covariant Voigt array conventional lab array

$$\begin{bmatrix} B_{11} \\ B_{22} \\ B_{33} \\ 2B_{(23)} \\ 2B_{(31)} \\ 2B_{(12)} \end{bmatrix} = \begin{bmatrix} Z_{111} & Z_{112} & Z_{311} \\ Z_{221} & Z_{222} & Z_{322} \\ Z_{331} & Z_{332} & Z_{333} \\ 2Z_{(23)1} & 2Z_{(23)2} & 2Z_{(23)3} \\ 2Z_{(31)1} & 2Z_{(31)2} & 2Z_{(31)3} \\ 2Z_{(12)1} & 2Z_{(12)2} & 2Z_{(12)3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \\ 0 & c_3 & c_2 \\ c_2 & 0 & c_1 \\ c_3 & c_1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{12.149a}$$

covariant Voigt matrix

and $B_I^{\mathrm{V}} = Z_{Ij}^{\mathrm{V}} u^j$, respectively. (12.149b)

Here, I is a Voigt free index ranging from 1 to 6, while j is a regular laboratory index implicitly summed from 1 to 3. The laboratory basis is orthonormal, so the notation u^j means the same thing as u_j, but superscripting is used here to conform to standard syntax that applies when any part of an indicial formula (in this case the Voigt part) does not correspond to an orthonormal basis.

In the contravariant Voigt representation, matrix and indicial forms of $\underset{\approx}{\mathbf{B}} = \underset{\approx}{\mathbf{Z}} \bullet \underset{\sim}{\mathbf{u}}$ are

contravariant Voigt array conventional lab array

$$\begin{bmatrix} B_{11} \\ B_{22} \\ B_{33} \\ B_{(23)} \\ B_{(31)} \\ B_{(12)} \end{bmatrix} = \begin{bmatrix} Z_{111} & Z_{112} & Z_{311} \\ Z_{221} & Z_{222} & Z_{322} \\ Z_{331} & Z_{332} & Z_{333} \\ Z_{(23)1} & Z_{(23)2} & Z_{(23)3} \\ Z_{(31)1} & Z_{(31)2} & Z_{(31)3} \\ Z_{(12)1} & Z_{(12)2} & Z_{(12)3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \\ 0 & \frac{c_3}{2} & \frac{c_2}{2} \\ \frac{c_2}{2} & 0 & \frac{c_1}{2} \\ \frac{c_3}{2} & \frac{c_1}{2} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{12.150a}$$

contravariant Voigt matrix

and $B_{\mathrm{V}}^{I} = Z_{\mathrm{V}}^{Ij} u_j$, respectively. (12.150b)

TENSOR ALGEBRA

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

In the Mandel representation, these forms of the operation $\underset{\approx}{\boldsymbol{B}} = \underset{\approx\!\sim}{\boldsymbol{Z}} \bullet \underset{\sim}{\boldsymbol{u}}$ are

Mandel array — conventional lab array — Mandel matrix

$$\begin{bmatrix} B_{\underline{11}} \\ B_{\underline{22}} \\ B_{\underline{33}} \\ B_{\underline{23}} \\ B_{\underline{31}} \\ B_{\underline{12}} \end{bmatrix} = \begin{bmatrix} Z_{\underline{11}1} & Z_{\underline{11}2} & Z_{\underline{31}1} \\ Z_{\underline{22}1} & Z_{\underline{22}2} & Z_{\underline{32}2} \\ Z_{\underline{33}1} & Z_{\underline{33}2} & Z_{\underline{33}3} \\ Z_{\underline{23}1} & Z_{\underline{23}2} & Z_{\underline{23}3} \\ Z_{\underline{31}1} & Z_{\underline{31}2} & Z_{\underline{31}3} \\ Z_{\underline{12}1} & Z_{\underline{12}2} & Z_{\underline{12}3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \\ 0 & \frac{c_3}{\sqrt{2}} & \frac{c_2}{\sqrt{2}} \\ \frac{c_2}{\sqrt{2}} & 0 & \frac{c_1}{\sqrt{2}} \\ \frac{c_3}{\sqrt{2}} & \frac{c_1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{12.151a}$$

and $B_{\underline{I}} = Z_{\underline{I}\,j}\, u_j$, respectively. (12.151b)

All of these matrix formulations are equivalent. Hence, it is up to your personal preferences which one to use. The Voigt system is more common in the literature, but (as previously mentioned) it is becoming increasingly seen as an unwise choice if you wish to apply conventional linear-algebra matrix operations like singular-value decomposition, as these make no sense for contravariant or covariant components of any tensor (Voigt or otherwise). The Mandel system provides components with respect to an orthonormal tensor basis and thus has no confusing distinction between contravariant and covariant components. *The Mandel system allows meaningful application of standard linear algebra decompositions that would be otherwise inappropriate when applied to Voigt matrices.*

<u>EXAMPLE 3</u>: Transformation of a tensor to a tensor $(V_3^2 \rightarrow V_3^2)$.

Consider a tensor-to-tensor function such that

input = ordinary engineering second-order tensor $\underset{\approx}{\boldsymbol{X}}$ (12.152a)

output = ordinary engineering second-order tensor $\underset{\approx}{\boldsymbol{Y}}$ (12.152b)

According to the representation theorem, there exists a local *tangent* fourth-order tensor $\underset{\approx\approx}{\boldsymbol{M}}$ that is independent of $d\underset{\approx}{\boldsymbol{X}}$ (but might depend on $\underset{\approx}{\boldsymbol{X}}$) such that $d\underset{\approx}{\boldsymbol{Y}} = \underset{\approx\approx}{\boldsymbol{M}}:d\underset{\approx}{\boldsymbol{X}}$. If the fourth-order tensor happens to be independent of $\underset{\approx}{\boldsymbol{X}}$, then the operation is linear so that

$$\underset{\approx}{\boldsymbol{Y}} = \underset{\approx\approx}{\boldsymbol{M}}:\underset{\approx}{\boldsymbol{X}} \quad \Leftrightarrow \quad Y_{ij} = M_{ijkl}X_{kl} \tag{12.153}$$

An assumption of linearity isn't really needed for the ensuing discussion, but it is adopted to keep the equations simpler (*i.e.,* the same concepts apply to *tangent* tensors).

TENSOR ALGEBRA

A very simple 9D matrix form of the above equation is

$$\begin{bmatrix} Y_{11} \\ Y_{22} \\ Y_{33} \\ Y_{23} \\ Y_{31} \\ Y_{12} \\ Y_{32} \\ Y_{13} \\ Y_{21} \end{bmatrix} = \begin{bmatrix} M_{1111} & M_{1122} & M_{1133} & M_{1123} & M_{1131} & M_{1112} & M_{1132} & M_{1113} & M_{1121} \\ M_{2211} & M_{2222} & M_{2233} & M_{2223} & M_{2231} & M_{2212} & M_{2232} & M_{2213} & M_{2221} \\ M_{3311} & M_{3322} & M_{3333} & M_{3323} & M_{3331} & M_{3312} & M_{3332} & M_{3313} & M_{3321} \\ M_{2311} & M_{2322} & M_{2333} & M_{2323} & M_{2331} & M_{2312} & M_{2332} & M_{2313} & M_{2321} \\ M_{3111} & M_{3122} & M_{3133} & M_{3123} & M_{3131} & M_{3112} & M_{3132} & M_{3113} & M_{3121} \\ M_{1211} & M_{1222} & M_{1233} & M_{1223} & M_{1231} & M_{1212} & M_{1232} & M_{1213} & M_{1221} \\ M_{3211} & M_{3222} & M_{3233} & M_{3223} & M_{3231} & M_{3212} & M_{3232} & M_{3213} & M_{3221} \\ M_{1311} & M_{1322} & M_{1333} & M_{1323} & M_{1331} & M_{1312} & M_{1332} & M_{1313} & M_{1321} \\ M_{2111} & M_{2122} & M_{2133} & M_{2123} & M_{2131} & M_{2112} & M_{2132} & M_{2113} & M_{2121} \end{bmatrix} \begin{bmatrix} X_{11} \\ X_{22} \\ X_{33} \\ X_{23} \\ X_{31} \\ X_{12} \\ X_{32} \\ X_{13} \\ X_{21} \end{bmatrix} \tag{12.154}$$

Though this *seems* compact, it admits no significant simplification in the special (and very common) situations for which input and/or output is symmetric. When the input is always symmetric, the fourth-order tensor may be presumed without loss to have **right minor symmetry** (meaning that $M_{ijkl} = M_{ijlk}$). When the output is symmetric, the fourth-order tensor has **left minor symmetry** (meaning $M_{ijkl} = M_{jikl}$). As in the preceding examples, the following 9D alternative representation naturally exploits symmetries:

Contravariant Voigt array (has no factors of 2)

Covariant Voigt array (has factors of2)

Contravariant Voigt matrix (has no factors of 2)

$$\begin{bmatrix} Y_{(11)} \\ Y_{(22)} \\ Y_{(33)} \\ Y_{(23)} \\ Y_{(31)} \\ Y_{(12)} \\ Y_{[32]} \\ Y_{[13]} \\ Y_{[21]} \end{bmatrix} = \begin{bmatrix} M_{(11)(11)} & M_{(11)(22)} & M_{(11)(33)} & M_{(11)(23)} & M_{(11)(31)} & M_{(11)(12)} & M_{(11)[32]} & M_{(11)[13]} & M_{(11)[21]} \\ M_{(22)(11)} & M_{(22)(22)} & M_{(22)(33)} & M_{(22)(23)} & M_{(22)(31)} & M_{(22)(12)} & M_{(22)[32]} & M_{(22)[13]} & M_{(22)[21]} \\ M_{(33)(11)} & M_{(33)(22)} & M_{(33)(33)} & M_{(33)(23)} & M_{(33)(31)} & M_{(33)(12)} & M_{(33)[32]} & M_{(33)[13]} & M_{(33)[21]} \\ M_{(23)(11)} & M_{(23)(22)} & M_{(23)(33)} & M_{(23)(23)} & M_{(23)(31)} & M_{(23)(12)} & M_{(23)[32]} & M_{(23)[13]} & M_{(23)[21]} \\ M_{(31)(11)} & M_{(31)(22)} & M_{(31)(33)} & M_{(31)(23)} & M_{(31)(31)} & M_{(31)(12)} & M_{(31)[32]} & M_{(31)[13]} & M_{(31)[21]} \\ M_{(12)(11)} & M_{(12)(22)} & M_{(12)(33)} & M_{(12)(23)} & M_{(12)(31)} & M_{(12)(12)} & M_{(12)[32]} & M_{(12)[13]} & M_{(12)[21]} \\ M_{[32](11)} & M_{[32](22)} & M_{[32](33)} & M_{[32](23)} & M_{[32](31)} & M_{[32](12)} & M_{[32][32]} & M_{[32][13]} & M_{[32][21]} \\ M_{[13](11)} & M_{[13](22)} & M_{[13](33)} & M_{[13](23)} & M_{[13](31)} & M_{[13](12)} & M_{[13][32]} & M_{[13][13]} & M_{[13][21]} \\ M_{[21](11)} & M_{[21](22)} & M_{[21](33)} & M_{[21](23)} & M_{[21](31)} & M_{[21](12)} & M_{[21][32]} & M_{[21][13]} & M_{[21][21]} \end{bmatrix} \begin{bmatrix} X_{(11)} \\ X_{(22)} \\ X_{(33)} \\ 2X_{(23)} \\ 2X_{(31)} \\ 2X_{(12)} \\ 2X_{[32]} \\ 2X_{[13]} \\ 2X_{[21]} \end{bmatrix} \tag{12.155}$$

where

$$X_{(ij)} = \frac{1}{2}(X_{ij} + X_{ji}), \qquad Y_{(ij)} = \frac{1}{2}(Y_{ij} + Y_{ji}) \tag{12.156a}$$

$$X_{[ij]} = \frac{1}{2}(X_{ij} - X_{ji}), \qquad Y_{[ij]} = \frac{1}{2}(Y_{ij} - Y_{ji}) \tag{12.156b}$$

$$M_{(ij)(kl)} = \frac{1}{4}(M_{ijkl} + M_{jikl} + M_{ijlk} + M_{jilk}) \tag{12.156c}$$

$$M_{(ij)[kl]} = \frac{1}{4}(M_{ijkl} + M_{jikl} - M_{ijlk} - M_{jilk}) \tag{12.156d}$$

$$M_{[ij](kl)} = \frac{1}{4}(M_{ijkl} - M_{jikl} + M_{ijlk} - M_{jilk}) \tag{12.156e}$$

$$M_{[ij][kl]} = \frac{1}{4}(M_{ijkl} - M_{jikl} - M_{ijlk} + M_{jilk}) \tag{12.156f}$$

This Voigt representation has a major disadvantage: the tensor basis that is paired with the Voigt components is not orthonormal, making *the tensor arrays different* on the two sides of Eq. (12.155) — the one on the right-hand-side of the equation has confusing factors of 2 (which are actually metrics) that are not present on the left side. Similar to what was done in the previous examples, the *Mandel* representation normalizes the Voigt basis to give the following more consistent structure:

Mandel form!

$$\begin{bmatrix} Y_{11} \\ Y_{22} \\ Y_{33} \\ Y_{\underline{23}} \\ Y_{\underline{31}} \\ Y_{\underline{12}} \\ Y_{\overline{32}} \\ Y_{\overline{13}} \\ Y_{\overline{21}} \end{bmatrix} = \left[\begin{array}{ccc:ccc:ccc}
M_{1111} & M_{1122} & M_{1133} & M_{11\underline{23}} & M_{11\underline{31}} & M_{11\underline{12}} & M_{11\overline{32}} & M_{11\overline{13}} & M_{11\overline{21}} \\
M_{2211} & M_{2222} & M_{2233} & M_{22\underline{23}} & M_{22\underline{31}} & M_{22\underline{12}} & M_{22\overline{32}} & M_{22\overline{13}} & M_{22\overline{21}} \\
M_{3311} & M_{3322} & M_{3333} & M_{33\underline{23}} & M_{33\underline{31}} & M_{33\underline{12}} & M_{33\overline{32}} & M_{33\overline{13}} & M_{33\overline{21}} \\ \hdashline
M_{\underline{23}11} & M_{\underline{23}22} & M_{\underline{23}33} & M_{\underline{23}\underline{23}} & M_{\underline{23}\underline{31}} & M_{\underline{23}\underline{12}} & M_{\underline{23}\overline{32}} & M_{\underline{23}\overline{13}} & M_{\underline{23}\overline{21}} \\
M_{\underline{31}11} & M_{\underline{31}22} & M_{\underline{31}33} & M_{\underline{31}\underline{23}} & M_{\underline{31}\underline{31}} & M_{\underline{31}\underline{12}} & M_{\underline{31}\overline{32}} & M_{\underline{31}\overline{13}} & M_{\underline{31}\overline{21}} \\
M_{\underline{12}11} & M_{\underline{12}22} & M_{\underline{12}33} & M_{\underline{12}\underline{23}} & M_{\underline{12}\underline{31}} & M_{\underline{12}\underline{12}} & M_{\underline{12}\overline{32}} & M_{\underline{12}\overline{13}} & M_{\underline{12}\overline{21}} \\ \hdashline
M_{\overline{32}11} & M_{\overline{32}22} & M_{\overline{32}33} & M_{\overline{32}\underline{23}} & M_{\overline{32}\underline{31}} & M_{\overline{32}\underline{12}} & M_{\overline{32}\overline{32}} & M_{\overline{32}\overline{13}} & M_{\overline{32}\overline{21}} \\
M_{\overline{13}11} & M_{\overline{13}22} & M_{\overline{13}33} & M_{\overline{13}\underline{23}} & M_{\overline{13}\underline{31}} & M_{\overline{13}\underline{12}} & M_{\overline{13}\overline{32}} & M_{\overline{13}\overline{13}} & M_{\overline{13}\overline{21}} \\
M_{\overline{21}11} & M_{\overline{21}22} & M_{\overline{21}33} & M_{\overline{21}\underline{23}} & M_{\overline{21}\underline{31}} & M_{\overline{21}\underline{12}} & M_{\overline{21}\overline{32}} & M_{\overline{21}\overline{13}} & M_{\overline{21}\overline{21}}
\end{array}\right] \begin{bmatrix} X_{11} \\ X_{22} \\ X_{33} \\ X_{\underline{23}} \\ X_{\underline{31}} \\ X_{\underline{12}} \\ X_{\overline{32}} \\ X_{\overline{13}} \\ X_{\overline{21}} \end{bmatrix} \tag{12.157}$$

The 9×9 matrix in this formula is the same as the contravariant matrix in Eq. (12.155) except that the last six rows and last six columns (i.e., the ones that are associated with off-diagonals) are multiplied by $\sqrt{2}$. Only the shaded parts are required in cases that the input and output are always symmetric tensors. Unlike Eq. (12.155), which needed factors of 2 in one of the 9×1 vector arrays (but not in the other one), the arrays for both $\underset{\approx}{\mathbf{X}}$ and $\underset{\approx}{\mathbf{Y}}$ are constructed in an identical way: the last six components are the same as the contravariant Voigt components except multiplied by $\sqrt{2}$. Specifically,

$$X_{\underline{ij}} = \sqrt{2}X_{(ij)}\,, \qquad Y_{\underline{ij}} = \sqrt{2}Y_{(ij)} \tag{12.158a}$$

$$X_{\overline{ij}} = \sqrt{2}X_{[ij]}\,, \qquad Y_{\overline{ij}} = \sqrt{2}Y_{[ij]} \tag{12.158b}$$

$$M_{ij\underline{kl}} = \sqrt{2}M_{ij(kl)}\,, \qquad M_{\underline{ij}kl} = \sqrt{2}M_{(ij)kl} \qquad M_{\underline{ij}\,\underline{kl}} = 2\,M_{(ij)(kl)} \tag{12.158c}$$

$$M_{ij\overline{kl}} = \sqrt{2}M_{ij[kl]}\,, \qquad M_{\underline{ij}\overline{kl}} = 2\,M_{(ij)[kl]} \tag{12.158d}$$

$$M_{\overline{ij}kl} = \sqrt{2}M_{[ij]kl}\,, \qquad M_{\overline{ij}\underline{kl}} = 2\,M_{[ij](kl)} \tag{12.158e}$$

$$M_{\overline{ij}\,\overline{kl}} = 2\,M_{[ij][kl]} \tag{12.158f}$$

The Mandel representation may seem strange at first with all of its factors of $\sqrt{2}$ or 2, but it is actually *much* easier to work with than a conventional Voigt representation for the following reasons:

- The *non-orthonormal* covariant basis that is paired with the 9×1 Voigt *contra*variant component array is the set of tensors with traditional 3×3 component matrices given by

  $$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}, \begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}$$

  See Exercise 12.3.

The *non-orthonormal* contravariant basis that is paired with the 9×1 Voigt *co*variant array is the set of tensors with traditional 3×3 component matrices given by

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\ \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\ \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix},\ \frac{1}{2}\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix},\ \frac{1}{2}\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix},\ \frac{1}{2}\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\ \frac{1}{2}\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix},\ \frac{1}{2}\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix},\ \frac{1}{2}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}$$

See Exercise 12.4.

The simple *orthonormal* basis that is paired with the 9×1 Mandel component array is the set of tensors with traditional 3×3 component matrices given by

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\ \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\ \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix},\ \frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix},\ \frac{1}{\sqrt{2}}\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix},\ \frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\ \frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix},\ \frac{1}{\sqrt{2}}\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix},\ \frac{1}{\sqrt{2}}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}$$

Equation 12.154 also has an orthonormal basis. Namely, it is given by

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\ \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\ \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix},\ \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix},\ \begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix},\ \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix},\ \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix},\ \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix},\ \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}$$

This basis, however lacks the advantage shared by both Voigt and Mandel bases of splitting both the input and output into symmetric and skew parts. For both the Mandel and Voigt system, the 9×9 transformation matrix will have its last three columns zero if the transformation depends only on $\text{sym}\underset{\approx}{\boldsymbol{X}}$; the first six columns will be zero if the transformation depends only on $\text{skw}\underset{\approx}{\boldsymbol{X}}$. The last three rows of either the Voigt or Mandel matrix will be zero if the output is symmetric; the first six rows will be zero if the output is skew. Thus, in these special and very common cases, the effective size of the transformation matrix is significantly reduced.

- Unlike the Voigt system, which has a factor of 2 in the 9×1 covariant (strain-like) array but no factor of 2 in the contravariant (stress-like) array, the 9×1 representations are the same for any tensor when using the Mandel system because *the Mandel system uses an orthonormal basis*. The factors of 2 in the Voigt system are actually metrics (resulting from the inner product of the covariant basis tensors with themselves). If the Voigt representation is adjusted so that both of the 9×1 arrays are of the same type (both covariant or both contravariant), the corresponding 9×9 transformation matrix would be of "mixed" form.
- The Mandel system has the appealing property that major symmetry of the fourth-order tensor ($M_{ijkl} = M_{klij}$) implies symmetry of the Mandel matrix ($M_{IJ} = M_{JI}$), whereas the mixed Voigt transformation matrices would not be symmetric.
- The identity operation, $\underset{\approx}{\boldsymbol{Y}} = \underset{\approx}{\boldsymbol{X}}$, has a transformation matrix that is simply the 9×9 identity matrix when using the Mandel system. Such is not the case for the Voigt system (see Exercise 12.7).
- The ordinary eigenvalues and eigenvectors of the Mandel 9×9 representation are meaningful. Such is not the case for the 9×9 Voigt representation, which requires a generalized eigenproblem involving the basis metrics. See Ref. [14].
- The Mandel matrix for the inverse of a fourth-order tensor is found by inverting the Mandel matrix for the fourth-order tensor. With the Voigt system, on the other hand, inverting the *contra*variant Voigt matrix will give you the *covariant* matrix for the inverse (converting this to the contravariant form requires multiplying the last three

rows and last three columns by a factor of 2, which is again a needlessly cumbersome consequence of the Voigt basis not being orthonormal.

Exercise 12.3 Confirm that the ordinary fully populated 3×3 component matrix,

$$\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix},$$ is indeed obtained by multiplying the nine contravariant components,

$\{Y_{11}, Y_{22}, Y_{33}, Y_{(23)}, Y_{(31)}, Y_{(12)}, Y_{[32]}, Y_{[13]}, Y_{[21]}\}$ times the corresponding nine covariant basis tensors having component matrices respectively given by

$$\begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&1\\0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\1&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\1&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&-1\\0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\-1&0&0 \end{bmatrix}, \begin{bmatrix} 0&-1&0\\1&0&0\\0&0&0 \end{bmatrix}$$

Hint: Multiply the first component times the first basis matrix plus the next component times the next basis matrix, etc.

Exercise 12.4 Confirm that the ordinary fully populated 3×3 ordinary component matrix,

$$\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix},$$ is indeed obtained by multiplying the nine covariant Voigt components,

$\{Y_{11}, Y_{22}, Y_{33}, 2Y_{(23)}, 2Y_{(31)}, 2Y_{(12)}, 2Y_{[32]}, 2Y_{[13]}, 2Y_{[21]}\}$ times the corresponding nine contravariant Voigt basis tensors having component matrices respectively given by

$$\begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0&0&0\\0&0&1\\0&1&0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0&0&1\\0&0&0\\1&0&0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0&1&0\\1&0&0\\0&0&0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0&0&0\\0&0&-1\\0&1&0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0&0&1\\0&0&0\\-1&0&0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0&-1&0\\1&0&0\\0&0&0 \end{bmatrix}$$

Hint: Same procedure as the previous problem.

Exercise 12.5 The operation $\underset{\sim}{v} = \underset{\sim}{A} \bullet \underset{\sim}{c}$ is said to be **bilinear** because it is linear with respect to both $\underset{\sim}{c}$ and $\underset{\sim}{A}$. As such, there exists a fourth-order tensor $\underset{\approx}{M}$ (independent of both $\underset{\sim}{c}$ and $\underset{\sim}{A}$) such that $v_i = A_{mn} M_{mniq} c_q$, which may be written in direct notation as $\underset{\sim}{v} = \underset{\sim}{A} : \underset{\approx}{M} \bullet \underset{\sim}{c}$. Find the Mandel component matrix for $\underset{\approx}{M}$ in the case that $\underset{\sim}{A}$ is a general, potentially nonsymmetric tensor. Find the simplest Mandel component matrix for $\underset{\approx}{M}$ in the case that $\underset{\sim}{A}$ is known to be always a skew tensor.

Hint: In the first case, explain why the answer is the general fourth-order identity (see previous exercises for the corresponding Mandel matrix). In the second case, you must impose, <u>without loss in generality</u>, a certain symmetry. Reasoning is similar to the symmetry imposed in Eq. (12.133).

Computer algorithm for evaluating 9x9 Mandel matrices. To compute the 9×9 Mandel matrix of a fourth-order tensor $\underset{\underset{\approx}{\approx}}{\boldsymbol{M}}$,

STEP 1. Find the conventional $3 \times 3 \times 3 \times 3$ representation, denoted M_{ijkl} in indicial notation.

STEP 2. Define a 9-member array whose elements are not numerals but instead are the 3×3 component matrices for the Mandel basis.

STEP 3. Loop over I and J each ranging from 1 to 9:

(i) Define $[U]$ to be the 3×3 component matrix of the I^{th} Mandel basis tensor (*i.e.*, the I^{th} element from step 2)

(ii) Define $[V]$ to be the 3×3 component matrix of the J^{th} Mandel basis tensor (*i.e.*, the J^{th} element from step 2).

(iii) Set M_{IJ} equal to $U_{ij} M_{ijkl} V_{kl}$, where the indices i, j, k, and l are each implicitly summed from 1 to 3.

The last two steps of this algorithm are implemented in *Mathematica* as

```
Mandel9by9[M3by3by3by3_] := Module[{U, V, MandelBasis = {
   {{1, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 1, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 1}},
   1/Sqrt[2] {{0, 0, 0}, {0, 0, 1}, {0, 1, 0}}, 1/Sqrt[2] {{0, 0, 1}, {0, 0, 0}, {1, 0, 0}}, 1/Sqrt[2] {{0, 1, 0}, {1, 0, 0}, {0, 0, 0}},
   1/Sqrt[2] {{0, 0, 0}, {0, 0, -1}, {0, 1, 0}}, 1/Sqrt[2] {{0, 0, 1}, {0, 0, 0}, {-1, 0, 0}}, 1/Sqrt[2] {{0, -1, 0}, {1, 0, 0}, {0, 0, 0}}
  }},
 Table[
  U = MandelBasis[[iman]]; V = MandelBasis[[jman]];
  Sum[U[[i, j]] M3by3by3by3[[i, j, k, l]] V[[k, l]], {i, 3}, {j, 3}, {k, 3}, {l, 3}]
  , {iman, 9}, {jman, 9}]
 ]
```

This is STEP 2 of the algorithm.

This is STEP 3.

EXAMPLE: suppose we wish to find the Mandel array for the fourth-order tensor defined by $\underset{\underset{\approx}{\approx}}{\boldsymbol{M}} = \underset{\sim}{\boldsymbol{a}}\underset{\sim}{\boldsymbol{b}}\underset{\approx}{\boldsymbol{C}}$ where the right-hand side of this equation is formed from dyadic multiplication of vectors and a tensor having laboratory components given by

$$\{\underset{\sim}{\boldsymbol{a}}\} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}, \qquad \{\underset{\sim}{\boldsymbol{b}}\} = \begin{bmatrix} 8 \\ 3 \\ 2 \end{bmatrix}, \qquad \text{and } [\underset{\approx}{\boldsymbol{C}}] = \begin{bmatrix} 4 & -5 & 0 \\ 0 & 1 & 7 \\ 2 & 3 & 0 \end{bmatrix}. \tag{12.159}$$

The conventional indicial formula for the desired fourth-order tensor is

$$M_{ijkl} = a_i b_j C_{kl} \tag{12.160}$$

To get the `16` component of the Mandel matrix, first define

$$[U] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 1^{\text{st}} \text{ Mandel basis matrix, and} \tag{12.161a}$$

$$[V] = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 6^{\text{th}} \text{ Mandel basis matrix.} \tag{12.161b}$$

Then only two of the 81 terms in the implied summation $U_{ij}M_{ijkl}V_{kl}$ survive:

$$\begin{aligned} M_{16} &= U_{ij}M_{ijkl}V_{kl} = U_{11}M_{1112}V_{12} + U_{11}M_{1121}V_{21} = a_1 b_1 C_{12}/\sqrt{2} + a_1 b_1 C_{21}/\sqrt{2} \\ &= (2)(8)(-5)/\sqrt{2} + (2)(8)(0)/\sqrt{2} = -40\sqrt{2} \end{aligned} \tag{12.162}$$

This tedious hand calculation verifies the circled `16` component in the following application of the previously defined automated Mathematica function:

```
In[2]:= a = {2, 0, -5}; b = {8, 3, 2}; ctensor = {{4, -5, 0}, {0, 1, 7}, {2, 3, 0}};
        M3by3by3by3 = Table[a[[i]] b[[j]] ctensor[[k, l]], {i, 3}, {j, 3}, {k, 3}, {l, 3}];
        Mandel9by9[M3by3by3by3] // Simplify // MatrixForm
```

Out[4]//MatrixForm=

$$\begin{pmatrix}
64 & 16 & 0 & 80\sqrt{2} & 16\sqrt{2} & -40\sqrt{2} & -32\sqrt{2} & -16\sqrt{2} & 40\sqrt{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-40 & -10 & 0 & -50\sqrt{2} & -10\sqrt{2} & 25\sqrt{2} & 20\sqrt{2} & 10\sqrt{2} & -25\sqrt{2} \\
-30\sqrt{2} & -\frac{15}{\sqrt{2}} & 0 & -75 & -15 & \frac{75}{2} & 30 & 15 & -\frac{75}{2} \\
-72\sqrt{2} & -18\sqrt{2} & 0 & -180 & -36 & 90 & 72 & 36 & -90 \\
12\sqrt{2} & 3\sqrt{2} & 0 & 30 & 6 & -15 & -12 & -6 & 15 \\
-30\sqrt{2} & -\frac{15}{\sqrt{2}} & 0 & -75 & -15 & \frac{75}{2} & 30 & 15 & -\frac{75}{2} \\
88\sqrt{2} & 22\sqrt{2} & 0 & 220 & 44 & -110 & -88 & -44 & 110 \\
-12\sqrt{2} & -3\sqrt{2} & 0 & -30 & -6 & 15 & 12 & 6 & -15
\end{pmatrix}$$

Exercise 12.6 Using the components defined in Eq. (12.159), find the 9×9 Mandel matrix for $\mathbf{M} = \mathbf{a}\mathbf{b}\mathbf{A}$ in which $\mathbf{A} = \text{sym}\mathbf{C}$. Provide at least one "by hand" verification calculation like what was done in Eq. (12.162). Then also find the Mandel Matrices for $\mathbf{T} = \mathbf{a}\mathbf{A}\mathbf{b}$ and $\mathbf{Y} = \mathbf{A}\mathbf{b}\mathbf{a}$.

Hint: This is like the example. The symmetry $A_{ij} = A_{ji}$ implies a new symmetry $(M_{ijkl} = M_{ijlk})$, which results in the Mandel matrix having its last three columns all zeros! What symmetries are evident for the other parts of this question?

TENSOR ALGEBRA

Exercise 12.7 Show at least two hand calculations (then use the computer) for the following tasks:

(a) Find the 9×9 Mandel and contravariant Voigt matrices for the identity operation, $\mathbf{Y} = \mathbf{X}$.

(b) Find the 9×9 Mandel and contravariant Voigt matrices for the symmetry operation, $\mathbf{Y} = \text{sym}\mathbf{X}$.

(c) Find the 9×9 Mandel and contravariant Voigt matrices for the skew operation, $\mathbf{Y} = \text{skw}\mathbf{X}$.

(d) Let $\mathbf{A}$ and $\mathbf{B}$ be known *symmetric* tensors. For the following operations, find the 9×9 Mandel matrix for the linear operator $\mathbf{M}$ such that $\mathbf{Y} = \mathbf{M}:\mathbf{X}$:

i. $\mathbf{Y} = \mathbf{A}(\mathbf{B}:\mathbf{X})$

ii. $\mathbf{Y} = \mathbf{A} \bullet \mathbf{X} \bullet \mathbf{B}$

For each, express the results in terms of the *Mandel* components of $\mathbf{A}$ and $\mathbf{B}$, written as $A_1, A_2, \ldots, A_9$ and $B_1, B_2, \ldots, B_9$, as defined in Eq. (12.126a).

(e) What revisions of the operation in part (*d*)*ii* would give a similar operation that produces both a major and minor symmetric fourth-order tensor? Resist peeking at the hint!

Hint: (a) Mandel's will be the simple identity matrix. (b) and (c) will be a "piece" of the identity matrix (if using the Mandel system) — this result nicely shows these to be projection operators! For part (d)i., note that symmetry of $\mathbf{B}$ *implies that* $\mathbf{B}:\mathbf{X} = \mathbf{B}:(\text{sym}\mathbf{X}) = B_1X_1 + \ldots + B_6X_6$ *(i.e., there is no involvement of* X_7, X_8, *or* X_9, *and hence the last three columns of the* 9×9 *matrix will be zero; ultimately, your answer will look like a higher-dimensional form of the component matrix for an ordinary vector-vector dyad* $\mathbf{a}\mathbf{b}$.

For part (d)ii., you need to first prove that $M_{ijkl} = A_{ik}B_{jl}$. *Then work out the matrix painstakingly for at least two components to show the required hand calculations. For example, the Mandel* M_{14} *component is* $M_{11\underline{23}} = \sqrt{2}M_{11(23)} = \frac{\sqrt{2}}{2}(M_{1123} + M_{1132}) = \frac{\sqrt{2}}{2}(A_{12}B_{13} + A_{13}B_{12})$. *This result now needs to be written in terms of the single-index Mandel components for* $\mathbf{A}$ *and* $\mathbf{B}$. *By virtue of symmetry,* $A_{12} = A_{\underline{12}} = A_6/\sqrt{2}$ *and* $B_{13} = B_{\underline{13}} = B_5/\sqrt{2}$. *The other components are similar. The final answer for the* M_{14} *component is therefore* $\frac{1}{2\sqrt{2}}(A_6B_5 + A_5B_6)$. *Once hand calculations like these have been given for two components, use the computer algorithm to get the rest. (e) The answer is to replace both the input and the output by their symmetric parts to get* $\mathbf{Y} = \text{sym}(\mathbf{A} \bullet \text{sym}\mathbf{X} \bullet \mathbf{B})$. *Your job is to confirm that this does indeed produce a 9x9 matrix that is nonzero only in the upper-left 6x6 submatrix!*

This concludes our overview of how to convert higher-order tensor expressions and operations to matrix forms. Such transformations are needed for actual computations (especially when built-in linear algebra computer packages are to be used). Throughout the remainder of this book, where the discussions are concept-focused, and direct notations and traditional indicial formulas (where indices range from 1 to 3) are typically sufficient, but even a passing understanding of what was covered in this section can significantly enhance your understanding of the upcoming survey of operations. Also, some of these operations are summarized in the Mandel system, which was why we had to introduce it here despite its abstract nature.